$$x \cdot y - x \cdot P(y) - y \cdot P(x) =$$

$$= ab\left(\|h\|^2 - \|f\|^2\right) - ab(h+f) \cdot P(h-f) - ab(h-f) \cdot P(h+f) =$$

$$= ab\left(\|h\|^2 - \|f\|^2 + \|f\|^2 + \|f\|^2\right) = ab\left(\|h\|^2 + \|f\|^2\right) = ab = \|x\| \, \|y\|$$

According to substitution $y := -y$ which transform inequality (1) to inequality (2) we obtain equality condition in inequality (2) : $x = a(h+f)$, $y = b(f-h)$, where a, b be arbitrary positive real numbers and $h \in F^1$, $f \in F$ with $\|h\|^2 + \|f\|^2 = 1$.
So, in inequality (3) equality occurs if $x = a(h+f)$, $y = b(h-f)$, where a, b be arbitrary real numbers and $h \in F^1$, $f \in F$ with $\|h\|^2 + \|f\|^2 = 1$.
Comment. Since $x \cdot P(y) = y \cdot P(x)$ inequality (3) can be rewritten in assymetric form

$$|x \cdot y - 2x \cdot P(y)| \leq \|x\| \, \|y\| \text{ or } |x \cdot y - 2y \cdot P(x)| \leq \|x\| \, \|y\|$$

W38. (Solution by the proposer.) We have the following:

$$\frac{4^x + 2^x + 1}{3} \geq \sqrt[3]{8^x} = 2^x > x$$

for $x \in R$, so

$$4^x + 2^x + 1 > 3x, \ \frac{4^x + 2^x + 1}{x} > 3 > 1$$

for all $x > 0$ and

$$\left(\frac{4^x + 2^x + 1}{x}\right)^x > 3^x > 2^x \text{ so } \left(\frac{4^x + 2^x + 1}{x}\right)^x - 2^x > 0$$

for all $x > 0$. We prove that

$$(1 + 2^x + 4^x) < x^x (1 + 2^x)$$

for all $x \in \left(0, \frac{1}{2e}\right)$. We take $f(x) = (1 + 2^x + 4^x)^x$ and $g(x) = x^x (1 + 2^x)$, when f is strivtly increasing function,

$$\frac{1}{6} < \frac{1}{2e} \Rightarrow \lim_{x\to 0} f(x) < f(x) \le f\left(\frac{1}{6}\right) < f\left(\frac{1}{2e}\right)$$

for all $x \in \left(0, \frac{1}{6}\right] \subset \left(0, \frac{1}{2e}\right]$ so

$$1 < f(x) \le f\left(\frac{1}{6}\right) < f\left(\frac{1}{2e}\right)$$

for all $x \in \left(0, \frac{1}{6}\right]$. We have $g(x) > 0,\ x \in \left(0, \frac{1}{2e}\right]$ and

$$g'(x) = x^x \ln(ex) + (2x)^x \ln(2ex),$$

$0 < ex < 2ex \le 1$ so

$$\ln(ex) < \ln(2ex) \le 0 \text{ and } g'(x) < 0$$

so g is strictly decreasing.

$$g\left(\frac{1}{2e}\right) \le g\left(\frac{1}{6}\right) \le g(x) < \lim_{x\to 0} g(x) = 2$$

so

$$f\left(\frac{1}{6}\right) < g\left(\frac{1}{6}\right) \le g(x) < 2$$

and $f(x) < g(x)$ for all $x \in \left(0, \frac{1}{2e}\right]$. Using the Bernoulli's inequality , we take

$$(1+x)^\alpha < 1 + \alpha x \text{ for } x > -1 \text{ and } \alpha \in (0,1)$$

so

$$6 + 6\sqrt[2]{2} + 6\sqrt[6]{4} < 22 < 64 < \left(\sqrt[6]{2} + 1\right)^6$$

etc.

Second solution. 1.

$$0 < \left(\frac{4^x + 2^x + 1}{x}\right)^x - 2^x \iff 2^x < \left(\frac{4^x + 2^x + 1}{x}\right)^x \iff 0 < 4^x + 2^x + 1 - 2x$$

Since $a^x = e^{x\ln a} > x\ln a + 1$,for $a > 1$ and $x > 0$ then

$$4^x + 2^x + 1 - 2x > 2x\ln 2 + 1 + x\ln 2 + 1 + 1 - 2x = (3\ln 2 - 2)x + 3 > 0$$

(because $x > 0$ and $2\sqrt{2} > 2 \cdot 1.4 = 2.8 > e \implies 8 > e^2 \implies 3\ln 2 > 2$).

2.

$$\left(\frac{4^x+2^x+1}{x}\right)^x - 2^x < 1 \iff \frac{4^x+2^x+1}{x} < (1+2^x)^{\frac{1}{x}}.$$

Since $\frac{1}{x} \geq 2e > 1$ then, applying inequality $(1+t)^{\alpha} \geq 1 + t\alpha + \binom{\alpha}{2}t^2$, w hich holds for any $t \geq 0$ and any $\alpha \geq 1$, to $t = 2^x, \alpha = \frac{1}{x}$ w e obtain

$$(1+2^x)^{\frac{1}{x}} \geq$$

$$\geq 1 + 2^x \cdot \frac{1}{x} + \frac{1}{2x}\left(\frac{1}{x}-1\right)(2^x)^2 = 1 + \frac{2^x}{x} + \frac{4^x(1-x)}{2x^2} > \frac{2^x}{x} + \frac{4^x(1-x)}{2x^2}.$$

Thus, suffices to prove

$$\frac{4^x+2^x+1}{x} < \frac{2^x}{x} + \frac{4^x(1-x)}{2x^2} \iff$$

$$4^x+2^x+1 < 2^x + \frac{4^x(1-x)}{2x} \iff 4^x + 1 < x + \frac{4^x(1-x)}{2x} \iff$$

$$\iff 4^x + 1 < \frac{4^x(1-x)}{2x} \iff 2 < 4^x\left(\frac{1}{x}-3\right).$$

Since $2e - 5 > 0 > 0$ and $4^x > 1$ then $\frac{1}{x} - 3 \geq 2e - 3 > 2$ and, therefore,

$$4^x\left(\frac{1}{x}-3\right) > \frac{1}{x} - 3 > 2.$$

Arkady Alt

W39. (Solution by the proposer.) We first prove that

$$\frac{\sqrt{a_i a_j}}{a_i + a_j} \leq \frac{\sqrt{x_i x_j}}{x_i + x_j}. \tag{1}$$

This is equivalent to

$$\frac{(a_i+a_j)^2}{a_i a_j} \geq \frac{(x_i+x_j)^2}{x_i x_j}.$$